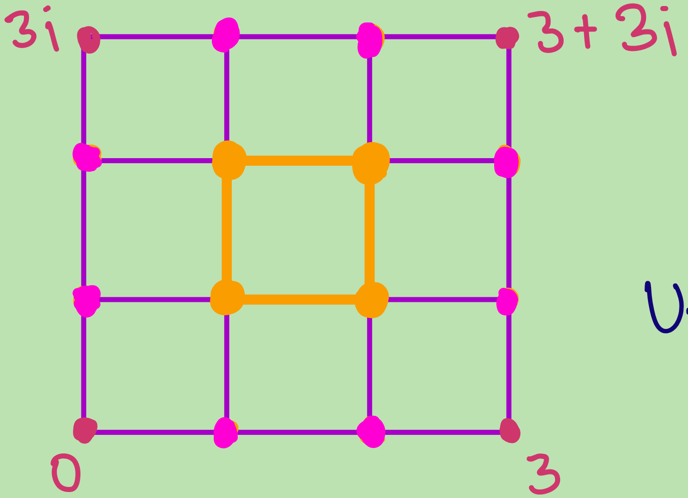
THE TOP H^* OF CONGRUENCE SUBGROUPS OF $SL_n R$ & $Sp_{2n} R$



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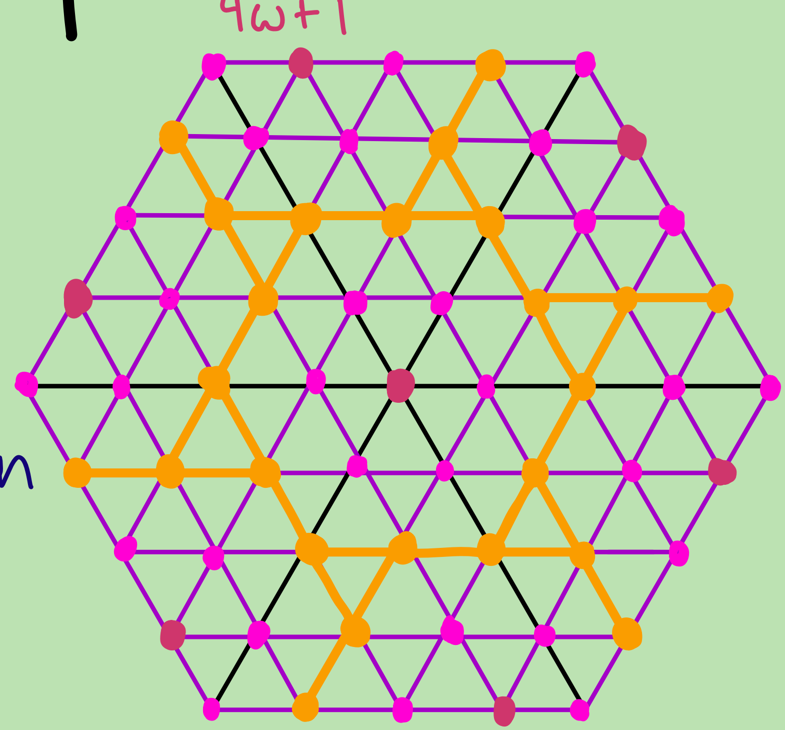


- **Examples of Arithmetic Groups:**

$SL_n R$, $Sp_{2n} R$

R : number ring

- **Principal Congruence subgroups:**

$\Gamma_n(p) \subset SL_n\mathbb{Z}$, $\Gamma_n(p) = \ker(SL_n\mathbb{Z} \to SL_n\mathbb{F}_p)$

R : Euclidean , $p \in R$ prime

$\Gamma_n(p) \subset SL_n R$, $\Gamma_n(p) = \ker(SL_n R \to SL_n(R/p))$

$\Gamma_{2n}^{sp}(p) \subset Sp_{2n} R$, $\Gamma_{2n}^{sp}(p) = \ker(Sp_{2n} R \to Sp_{2n}(R/p))$

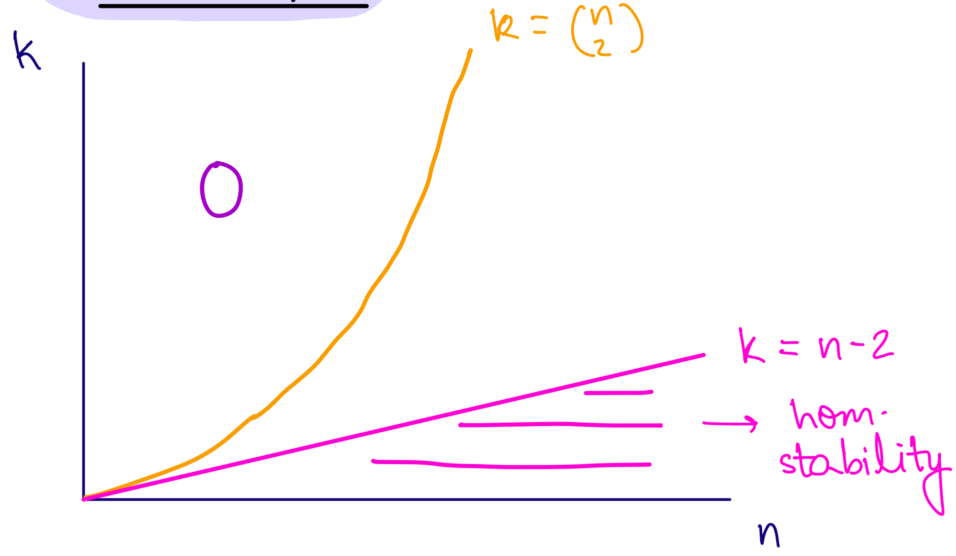
- **$H^*(\Gamma_n(p))$** : via symmetric spaces

$\Gamma_n(p) \subset SL_n\mathbb{Z}$

$\Gamma_n(p) \curvearrowright SL_n\mathbb{R}/SO(n) = X$

$\underline{H^*(\Gamma_n(p);\mathbb{Q}) \cong H^*(X/\Gamma_n;\mathbb{Q})}$

- **$H^k(\Gamma_n(p))$**



- **Thm [Borel – Serre]**

R number ring , $\Gamma <_{f.i.} SL_n R$

$H^{\nu-i}(\Gamma;\mathbb{Q}) \cong H_i(\Gamma;\mathcal{D})$

dualizing module

$\nu = \nu(R)$: quadratic in n

Goal: Compute $H^{top}(\Gamma) = H^{\nu}(\Gamma)$

$\cong H_0(\Gamma;\mathcal{D}) \cong (\mathcal{D})_\Gamma$

Computations

$SL_n(R)$:

$R = \mathbb{Z}$	(Lee - Szczarba) $\operatorname{rk} H^{top}(\Gamma_n(3)) = 3^{\binom{n}{2}}$	(Miller-Patzt-Putman) $\operatorname{rk} H^{top}(\Gamma_n(5)) > 2^{n-1} 5^{\binom{n}{2}}$ recursive in n...

Thm [P.]:

$R=\mathbb{Z}[i]$	$\operatorname{rk} H^{top}(\Gamma_n(2i+1)) = 5^{\binom{n}{2}}$	$\operatorname{rk} H^{top}(\Gamma_n(3)) > 2^{n-1} 9^{\binom{n}{2}}$ recursive in n...
$R=\mathbb{Z}[\omega]$	$\operatorname{rk} H^{top}(\Gamma_n(3\omega+1)) = 7^{\binom{n}{2}}$	$\operatorname{rk} H^{top}(\Gamma_n(4\omega+1)) > 2^{n-1} 13^{\binom{n}{2}}$ recursive in n...
$R=\mathbb{Z}[\sqrt{2}]$	$\operatorname{rk} H^{top}(\Gamma_n(3+\sqrt{2})) = 7^{\binom{n}{2}}$	$\operatorname{rk} H^{top}(\Gamma_n(5)) > 2^{n-1} 25^{\binom{n}{2}}$ recursive in n...
$R=\mathbb{Z}[\xi_5]$	$\operatorname{rk} H^{top}(\Gamma_n(2+\xi_5+\xi_5^2)) = 11^{\binom{n}{2}}$	$\operatorname{rk} H^{top}(\Gamma_n(3)) > 2^{n-1} 81^{\binom{n}{2}}$ recursive in n...

$Sp_{2n}(R)$:

Thm [P.]:

$R = \mathbb{Z}$	$\operatorname{rk} H^{top}(\Gamma_n^{sp}(3)) = 3^{n^2}$
$R = \mathbb{Z}[i]$	$\operatorname{rk} H^{top}(\Gamma_n^{sp}(2i+1)) = 5^{n^2}$
$R = \mathbb{Z}[\omega]$	$\operatorname{rk} H^{top}(\Gamma_n^{sp}(3\omega+1)) = 7^{n^2}$
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$R = \mathbb{Z}[\xi_5]$	$\operatorname{rk} H^{top}(\Gamma_n^{sp}(2+\xi_5+\xi_5^2)) = 11^{n^2}$

The General Pattern

$\mathbb{F} = R/(p)$

R	p	$\operatorname{rk} H^{top}$	$\lvert\mathbb{F}\rvert$	$R^\times$	$\lvert\mathbb{F}^\times/U\rvert$
$\mathbb{Z}$	3	$3^{\binom{n}{2}}$	3	$\{\pm 1\}$	1
$\mathbb{Z}[i]$	$2i+1$	$5^{\binom{n}{2}}$	5	$\{\pm 1, \pm i\}$	1
$\mathbb{Z}[\omega]$	$3\omega+1$	$7^{\binom{n}{2}}$	7	$\{\pm 1, \pm\omega, \pm\omega^2\}$	1
$\mathbb{Z}[\sqrt{2}]$	$3+\sqrt{2}$	$7^{\binom{n}{2}}$	7	$\{\pm 1\} \times \{(1+\sqrt{2})^k\}$	1
$\mathbb{Z}[\zeta_5]$	$2+\zeta_5+\zeta_5^2$	$11^{\binom{n}{2}}$	11	$\{\pm \zeta_5^\ell\} \times \{(\frac{1+\sqrt{5}}{2})^k\}$	1

$U = \operatorname{im}(R^\times \to \mathbb{F}^\times)$

The General Pattern

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R	p	$\mathrm{rk}\, H^{top}$	$\lvert\mathbb{F}\rvert$	$R^\times$	$\lvert\mathbb{F}^\times/U\rvert$
$\mathbb{Z}$	5	$> 2^{n-1}\, 5^{\binom{n}{2}}$	5	$\{\pm 1\}$	2
$\mathbb{Z}[i]$	3	$> 2^{n-1}\, 9^{\binom{n}{2}}$	9	$\{\pm 1, \pm i\}$	2
$\mathbb{Z}[\omega]$	$4\omega+1$	$> 2^{n-1}\, 13^{\binom{n}{2}}$	13	$\{\pm 1, \pm\omega, \pm\omega^2\}$	2
$\mathbb{Z}[\sqrt{2}]$	5	$> 2^{n-1}\, 25^{\binom{n}{2}}$	25	$\{\pm 1\} \times \{(1+\sqrt{2})^k\}$	2
$\mathbb{Z}[\zeta_5]$	3	$> 2^{n-1}\, 81^{\binom{n}{2}}$	81	$\{\pm \zeta_5^\ell\} \times \left\{\left(\frac{1+\sqrt{5}}{2}\right)^k\right\}$	2

$SL_n(R)$:

Thm [P.] : R Euclidean number ring
$p \in R$ prime, $\mathbb{F} = R/(p)$
$U = \mathrm{im}(R^\times \longrightarrow \mathbb{F})$

- $|\mathbb{F}^\times / U| = 1$ or 2

$\Rightarrow$ combinatorial description of $H^{top}(\Gamma_n(p))$

$Sp_{2n}(R)$:

Thm [P.] : R Euclidean number ring
$p \in R$ prime, $\mathbb{F} = R/(p)$
$U = \mathrm{im}(R^\times \longrightarrow \mathbb{F})$

- $|\mathbb{F}^\times / U| = 1$

$\Rightarrow$ combinatorial description of $H^{top}(\Gamma_{2n}^{sp}(p))$

Next :
- What is this combinatorial description?
- Flavour of proof.

- **Thm [Borel – Serre]**

  R number ring, $\Gamma <_{f.i.} SL_n R$

  $$H^{\nu-i}(\Gamma;\mathbb{Q}) \cong H_i(\Gamma;\mathcal{D})$$

  dualizing module

  $\nu = \nu(R)$: quadratic in n

- $H^{top}(\Gamma) = H^{\nu}(\Gamma) \cong H_0(\Gamma;\mathcal{D})$

  $\cong (\mathcal{D})_{\Gamma} \cong \mathcal{D}/\langle d-\gamma d \mid \gamma\in\Gamma\rangle$

- **$\mathcal{D}$: Steinberg Module**

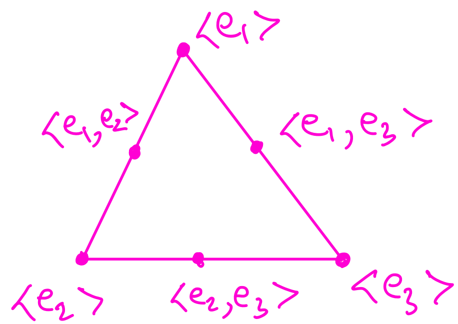
  $St_n R := \tilde{H}_{n-2}(T_n R)$

  Tits building

- Vertices of $T_n R$: $0 \subsetneq W \subsetneq R^n$ summands

  k-simplices : $0 \subsetneq W_0 \subsetneq \dots \subsetneq W_k \subsetneq R^n$

  $n=3$: Subcplex of $T_3 R$



  $T_3 R$ built by "gluing such spheres together"

- $T_n(R) \simeq \bigvee S^{n-2}$

  $St_n(R) := \tilde{H}_{n-2}(T_n(R))$

  Note: $SL_n R \curvearrowright T_n R$

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Goal : Compute $(St_n R)_{\Gamma_n(P)}$

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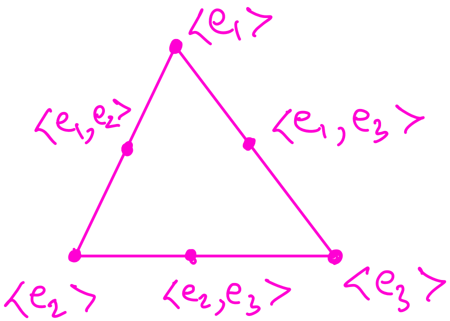

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- <u>[Lee – Szczarba]</u>: For $\Gamma_n(3) \subset SL_n \mathbb{Z}$,

  $(St_n \mathbb{Z})_{\Gamma_n(3)} \cong St_n \mathbb{F}_3$

- <u>[Miller - Patzt - Putman]</u>: For $\Gamma_n(5) \subset SL_n \mathbb{Z}$,

  $(St_n \mathbb{Z})_{\Gamma_n(5)} \cong St_n^{\pm} \mathbb{F}_5$

- <u>Big Idea</u> :
  - Construct presentations for $St_n \mathbb{F}_3$, $St_n^{\pm} \mathbb{F}_5$
  - Compare with $(St_n \mathbb{Z})_{\Gamma_n(P)}$

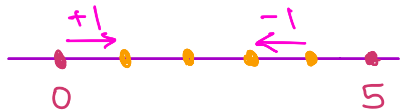
- Construct Presentations using high-connectivity of certain simplicial complexes

- The restrictions of $|\mathbb{F}^{\times}/U| = 1$ or 2 allow for high-connectivity results

Consequences of $|\mathbb{F}^\times / U| = 2$:

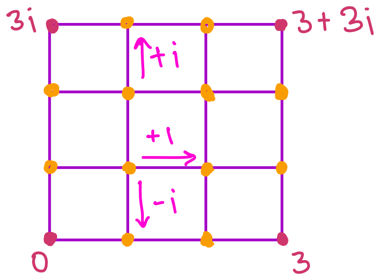
• $\mathbb{F}$ is "additively connected" by U

$R = \mathbb{Z}$, $p = 5$



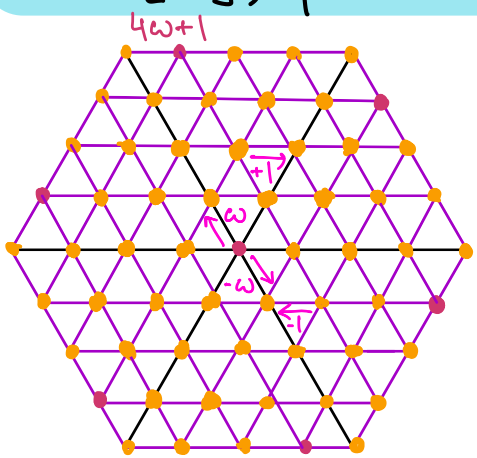
$U = \{\pm 1\}$

$R = \mathbb{Z}[i]$, $p = 3$



$U = \{\pm 1, \pm i\}$

$R = \mathbb{Z}[\omega]$, $p = 4\omega + 1$



$U = \{\pm 1, \pm \omega, \pm \omega^2\}$

• $\mathbb{F}^\times \setminus U$ is "additively connected" by U

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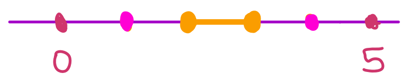

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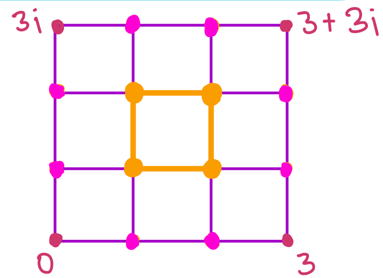

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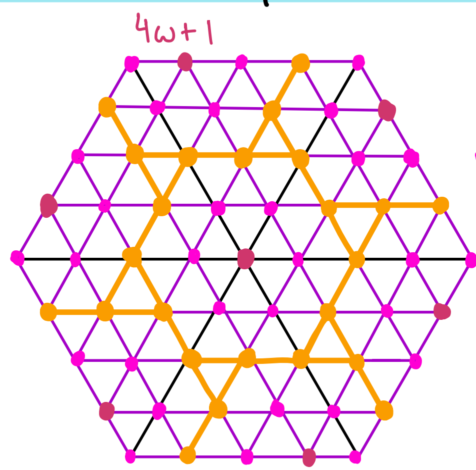

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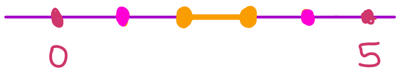

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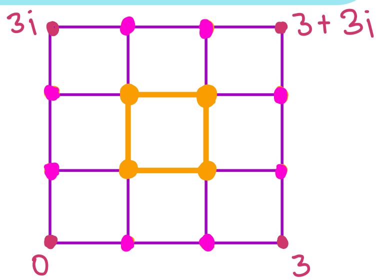

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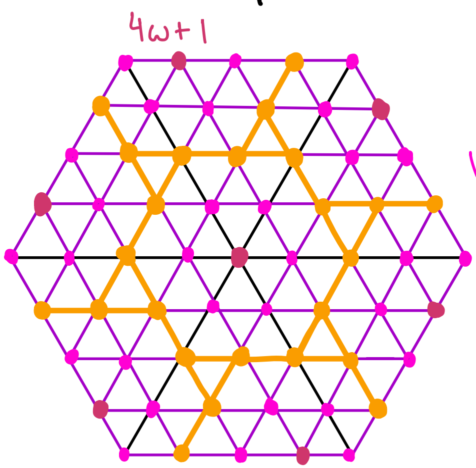

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- The restrictions of $|\mathbb{F}^\times / U| = 1$ or 2 allow for high-connectivity results

Thm [P.]: R Euclidean number ring
$p \in R$ prime, $\mathbb{F} = R/(p)$
$U = \operatorname{im}(R^\times \longrightarrow \mathbb{F})$

- $|\mathbb{F}^\times / U| = 1 \Rightarrow H^{top}(\Gamma_n(p)) \cong St_n \mathbb{F}$

- $|\mathbb{F}^\times / U| = 2 \Rightarrow H^{top}(\Gamma_n(p)) \cong St_n^U \mathbb{F}$

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$\mathrm{Sp}_{2n} R$

- Similar duality story, with
  symplectic Tits building $T_{2n}^{sp} R$
  symplectic steinberg module $St_{2n}^{sp} R$

- **Thm [P.]:** Have an exact sequence of sympl. & non-sympl. steinberg modules

$$0 \to St_{2n}\mathbb{F} \to \bigoplus_{\substack{W \subset \mathbb{F}^{2n} \\ W \text{ sympl.} \\ \dim = 2}} St(W^\perp) \otimes St^{sp}(W) \to \dots \to St_{2n}^{sp}\mathbb{F} \to 0$$

- **Thm [P.]:** R Euclidean number ring
$p \in R$ prime, $\mathbb{F} = R/(p)$
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Then $|\mathbb{F}^\times / U| = 1 \Rightarrow H^{top}(\Gamma_{2n}^{sp}(p)) \cong St_{2n}^{sp}\mathbb{F}$

$Sp_{2n} R$

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- **Thm [P.]**: R Euclidean number ring $p \in R$ prime, $\mathbb{F} = R/(p)$ $U = \operatorname{im}(R^{\times} \to \mathbb{F})$

Then $|\mathbb{F}^{\times}/U| = 1 \Rightarrow H^{top}(\Gamma_{2n}^{sp}(p)) \cong St_{2n}^{sp}\mathbb{F}$

Questions

1. What about the $|\mathbb{F}^{\times}/U| = 2$ case for $Sp_{2n}R$?
2. What happens when $\mathbb{F}^{\times}/U$ is bigger?
3. What about Chevalley groups of other types? (Eg: $SO_{n,n}$, $SO_{n,n+1}$)

THANK YOU!